STEIN NEIGHBORHOODS OF GRAPHS OF HOLOMORPHIC MAPPINGS

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Abstract. In this paper we provide sufficient conditions for the graphs of holomorphic mappings on compact sets in complex manifolds to have Stein neighborhoods. We show that under these conditions the mappings have properties analogous to properties of holomorphic functions on compact sets in $\mathbb{C}^n$.

1. Introduction

Let N and M be complex manifolds and let K be a compact set in N. There are several different ways to define holomorphic mappings between K and M but whatever definition is chosen there are at least three questions we would like to answer:

(1) Can any such mapping be approximated by holomorphic mappings on neighborhoods of K?
(2) If two such mappings are close, are they homotopic to each other in the space of holomorphic mappings on K?
(3) Is there a flexible way to shift these mappings?

These questions are either well-studied or trivial when M and N are Euclidean spaces. One can easily mimic this situation when K and its image have Stein neighborhoods by imbedding these neighborhoods into Euclidean spaces. But, in general, neither K nor its image need to have Stein neighborhoods. For example, they may contain compact complex submanifolds. A more exquisite example of this kind due to Rosay, when a mapping of a closed unit ball into an Euclidean space is an embedding of the open unit ball, can be found in [For].

If we define holomorphic mappings on K as restrictions of holomorphic mappings on its neighborhoods and assume that K has a basis of Stein neighborhoods, then we can look at the graph of the mapping on such a neighborhood U. Since the graph is a Stein complex submanifold of $U \times M$ by Siu's theorem ([S]) we see that the graph has a Stein neighborhood. After that we can proceed as above.

But if we give more reasonable definitions for holomorphic mappings on compact sets, then the situation is much more subtle. In [DrnFor] and [For] one can find an excellent discussion of arising problems and there solutions in many cases. However, finally, everything is reduced to the question: When the graph of a holomorphic mapping has a basis of Stein neighborhoods? And the goal of this paper is to provide a partial but usable answer to this question and also the answers to three questions above.

2000 *Mathematics Subject Classification.* Primary: 32Q28; secondary: 32H02, 32E30.
Key words and phrases. Plurisubharmonic functions, pluripotential theory.
The author was supported by the NSF Grant DMS-0900877.

The author is grateful to Franc Forstnerič for fruitful and inspiring discussions.

2. Fusion

In the future we will need to fuse plurisubharmonic functions. This means that given a finite collection of open sets U_j and plurisubharmonic functions u_j on U_j we want to find a plurisubharmonic function u on the union U of all U_j whose values are comparable with values of u_j. In general, fusion is impossible. For example, one can cover the infinite hyperplane in $\mathbb{CP}^n$ by open sets and define plurisubharmonic functions on these sets equal to $-\infty$ exactly on the hyperplane, but there are no plurisubharmonic functions defined on a neighborhood of the infinite hyperplane and equal to $-\infty$ on the hyperplane.

Analyzing the proof of Theorem 1 about approximations of plurisubharmonic functions in [FW] P. Gauthier in [Gau] came to a way to fuse two (pluri)subharmonic functions on domains in $(\mathbb{C}^n)\mathbb{R}^n$. This method was modified by N. Gogus in [Go] to fuse finitely many functions. The main ingredient of the proof in [Gau] was the existence of a strictly (pluri)subharmonic function on $(\mathbb{C}^n)\mathbb{R}^n$.

We use this idea to find a more delicate way of fusing. For this we introduce a *fusing device* which consists of a finite collection of triples (V_j, U_j, χ_j), $1 \le j \le k$, where $V_j \subset\subset U_j$ are open sets in a complex manifold N and χ_j is a C^∞-function on N with compact support in U_j, equal to 1 on V_j and taking values between 0 and 1 elsewhere and a strictly plurisubharmonic function δ on N taking values between 0 and 1. Let us take the maximal $\lambda > 0$ such that the functions $w_j = \delta + \lambda\chi_j$ are plurisubharmonic on N for all j. We define the fusing norm as $c = 3 + 2\lambda^{-1}$.

The advantage of our fusing result is that the same fusing device can fuse plurisubharmonic functions defined on different neighborhoods.

Theorem 2.1. *Let P be a holomorphic mapping of a complex manifold M into a complex manifold N. Suppose that $\{\{(V_j, U_j, \chi_j), 1 \le j \le k\}, \delta\}$ is a fusing device on N and c is its fusing norm.*

Let K be a compact set in M such that $K' = P(K) \subset \cup_{j=1}^k V_j$. Suppose also that there are open sets $W_j \subset N$, $1 \le j \le k$, such that $U_j \cap K' \subset W_j$ and $K' \subset \cup_{j=1}^k W_j$ and plurisubharmonic functions u_j on $\widetilde{W}_j = P^{-1}(W_j)$ never taking $-\infty$ as their values. Then there are a neighborhood $Z \subset \cup_{j=1}^k W_j$ of K' and a plurisubharmonic function u on $\widetilde{Z} = P^{-1}(Z)$ of K such that

$$\|u - u_j\|_{\widetilde{W}_j \cap \widetilde{Z}} \le c \max\{\|u_l - u_m\|_{\widetilde{W}_l \cap \widetilde{W}_j}, 1 \le m \le l \le k\}.$$

Proof. Let

$$b = \max\{\|u_l - u_m\|_{\widetilde{W}_l \cap \widetilde{W}_j}, 1 \le m \le l \le k\}.$$

If $b = 0$ then we can take $u(z) = u_j(z)$ if $z \in \widetilde{W}_j$. If $b = \infty$ then we can take as u any plurisubharmonic function on M, for example, $\delta \circ P$. So we assume that $0 < b < \infty$.

Let $v_j(z)$ be functions on M equal to $u_j(z) + 2\lambda^{-1} b w_j(P(z))$ on $\widetilde{W}_j \cap \widetilde{U}_j$, where $\widetilde{U}_j = P^{-1}(U_j)$, and to $-\infty$ elsewhere. On $\widetilde{W} = \cup_{j=1}^k (\widetilde{W}_j \cap \widetilde{U}_j)$ we define the function

$$u(z) = \max\{v_j(z) : 1 \le j \le k\}. \qquad (1)$$

Let us show that u is plurisubharmonic on a neighborhood $\widetilde{Z}$ of K. If $z \in \widetilde{W}$ does not belong to the boundary of any set $\widetilde{W}_j \cap \widetilde{U}_j$, then u is plurisubharmonic near z.

Now suppose that $z \in K$ and the set $L = \{j_1, \dots, j_p\}$ of indexes such that $z \in \partial(\widetilde{W}_{j_l} \cap \widetilde{U}_{j_l})$ for some $j_l \in L$ is non empty. If $j_l \in L$ then $\zeta = P(z) \in \partial U_{j_l}$ because $K' \cap U_{j_l} \subset W_{j_l}$. Hence, $\chi_{j_l} \equiv 0$ on the intersection of some neighborhood Q_{j_l} of ζ with $W_{j_l} \cap U j_l$.

There is j such that $\zeta = P(z) \in V_j \cap K'$. We may assume that $\chi_j \equiv 1$ on Q_{j_l}. Consequently, for $w \in P^{-1}(Q_{j_l} \cap W_{j_l} \cap U_{j_l})$ and $\xi = P(w)$ we have

$$v_{j_l}(w) - v_j(w) = u_{j_l}(w) - u_j(w) - 2b(\chi_j(\xi) - \chi_{j_l}(\xi)) \leq -b < 0.$$

Hence $v_j(w) > v_{j_l}(w)$.

Now let Q' be the intersection of all Q_{j_l}, $j_l \in L$, and let $Q \subset Q'$ be a neighborhood of ζ such that $Q \cap \partial(W_l \cap U_l) = \emptyset$ for $l \notin L$. If $z \in P^{-1}(Q)$ then

$$u(z) = \max\{v_j(z) : j \notin L\}$$

and this means that u is plurisubharmonic on $P^{-1}(Q)$. Therefore, there is a neighborhood $Z \subset \cap_{j=1} W_j$ of K' such that u is plurisubharmonic on $\widetilde{Z}$.

If $z \in \widetilde{W}_j \cap \widetilde{Z}$ then there is l between 1 and k such that

$$u(z) - u_j(z) = v_l(z) - u_j(z) = u_l(z) - u_j(z) + 2b(\lambda^{-1}\delta(P(z)) + \chi_j(P(z))).$$

Hence

$$\|u - u_j\|_{\widetilde{W}_j \cap \widetilde{Z}} \leq b(3 + 2\lambda^{-1}).$$

□

This theorem can be used to construct strictly pseudoconvex functions on M. Let us recall (see [Nar]) that an upper semicontinuous function u on a complex manifold M is called *strictly plurisubharmonic* if for every $z \in M$ there is a neighborhood V and a C^∞ strictly plurisubharmonic function v such that $u - v$ is plurisubharmonic on V.

Corollary 2.2. *Suppose, additionally, that in assumptions of Theorem 2.1 the fibers $P^{-1}(z)$ are manifolds for all $z \in N$ and the functions u_j are smooth and strictly plurisubharmonic on $P^{-1}(z) \cap \widetilde{W}_j$ for all $z \in W_j$. Then there is a neighborhood V of K and a strictly plurisubharmonic function on V.*

Proof. For the proof we note that the functions v_j in the proof of Theorem 2.1 are smooth and and strictly plurisubharmonic on $P^{-1}(z) \cap \widetilde{W}_j$ for every $z \in \widetilde{W}_j$. Therefore, the functions $v_j(z) + \delta(P(z))$ are strictly plurisubharmonic on $\widetilde{W}_j$. If a function u is defined by (1) then

$$u(z) + \delta(P(z)) = \max\{v_j(z) + \delta(P(z)) : 1 \leq j \leq k\}$$

is strictly plurisubharmonic on a neighborhood $\widetilde{Z}$ of K as the maximum of finitely many strictly plurisubharmonic functions. □

3. Stein neighborhoods

Let K be a compact set in a complex manifold N and let M be a complex manifold. We denote by $A(K,M)$ the set of all continuous mappings of K into M which are holomorphic in the interior of K. Let $H(K,M)$ be the set of all continuous mappings of K into M which can be uniformly approximated with any precision by holomorphic mappings of neighborhoods of K into M and let $H_{loc}(K,M)$ be the set of all continuous mappings of K into M with the following property: for any $f \in H_{loc}(K,M)$ and any point $z \in K$ there is a neighborhood U of z such that f can be uniformly approximated with any precision on $K \cap \overline{U}$ by mappings into M holomorphic on neighborhoods of $K \cap \overline{U}$. Clearly, $H(K,M) \subset H_{loc}(K,M) \subset A(K,M)$.

The famous Mergelyan theorem states that if $N = \mathbb{C}$ then $H(K,\mathbb{C}) = A(K,\mathbb{C})$. In several variables such a theorem does not hold, for example, when K is a closed disk in $\mathbb{C}^2$ since K has no interior. Until a better definition for $A(K,M)$ will show up an analog of the Mergelyan theorem can be searched only when K is the closure of its interior (see [GoP]).

Following the classical theory we continue to say that $A(K,M)$ has the *Mergelyan property* if $A(K,M) = H(K,M)$ and introduce the *local Mergelyan property* claiming that $A(K,M) = H_{loc}(K,M)$. Clearly, $A(K,M)$ has the local Mergelyan property if $A(K,\mathbb{C})$ has it. For example, if K is the closure of any domain in N with C^1-boundary or any compact set in a Riemann surface, then $A(K,M)$ has the local Mergelyan property.

The problem of Stein neighborhoods for graphs of holomorphic mappings is well discussed in [For], where it was proved that if K is the closure of a strongly pseudoconvex domain in M and $f \in A(K,M)$, then the graph of f on K has a basis of Stein neighborhoods. A reader can also find there counterexamples which show that such a statement fails in general settings.

In this paper Forstnerič poses the following problem: Let K be a compact set with a Stein neighborhood basis in a complex manifold N. Assume that $f \in H(K,M)$. Does the graph of f on K admits a basis of open Stein neighborhoods in $N \times M$? The following theorem gives an affirmative answer to this question even in more general form.

Theorem 3.1. *Suppose that a compact set K in a complex manifold N has a basis of Stein neighborhoods. If M is a complex manifold of dimension m and $f \in H_{loc}(K,M)$, then the graph of f on K has a basis of Stein neighborhoods in $N \times M$.*

Proof. If $f : A \to B$ is a mapping and $C \subset A$, then Γ^f_C will denote the graph of f on C.

The proof will follow in several steps. At the first step we select an appropriate covering of Γ^f_K. For this we pick up a Riemann metric ρ on M. For every point $z \in M$ there is its neighborhood B_z, a biholomorphic mapping ϕ_z of B_z onto the unit ball in $\mathbb{C}^m$ and a constant $a_z > 1$ such that

$$a_z^{-1}\|\phi_z(z_1) - \phi_z(z_2)\| \le \rho(z_1,z_2) \le a_z\|\phi_z(z_1) - \phi_z(z_2)\|.$$

Then we take a neighborhood $X' \subset N \times M$ of Γ^f_K and find a Stein neighborhood $Y \subset\subset N$ of K and a continuous extension $\widetilde{f}$ of f to $\overline{Y}$ such that $\Gamma^{\widetilde{f}}_{\overline{Y}} \subset X'$ and there is a strictly plurisubharmonic function δ on Y with values between 0 and 1

on Y. We cover $\widetilde{f}(\overline{Y})$ by finitely many neighborhoods $B_{z_j} = B_j$, $1 \le j \le k$, and let $\phi_j = \phi_{z_j}$ and $a = \max_{1 \le j \le k} a_{z_j}$. If $z_1, z_2 \in B_j \cap B_l$ then

$$a^{-2}\|\phi_j(z_1) - \phi_j(z_2)\| \le \|\phi_l(z_1) - \phi_l(z_2)\| \le a^2\|\phi_j(z_1) - \phi_j(z_2)\|.$$

For every point $\zeta \in K$ there are its neighborhood $U_\zeta \subset Y$ and a set $B_j = B_{j\zeta}$ such that $\Gamma^{\widetilde{f}}_{U_\zeta} \subset\subset U_\zeta \times B_j \subset X'$. Moreover, we may assume that f can be uniformly approximated with any precision on $K \cap \overline{U}_\zeta$ by mappings holomorphic on neighborhoods of $K \cap \overline{U}_\zeta$. We choose a domain $V_\zeta \subset\subset U_\zeta$ containing ζ and find a finite covering of K by open sets $V_j = V_{\zeta_j}$, $1 \le j \le p$. We set $U_j = U_{\zeta_j}$ and $B_j = B_{j\zeta}$.

Let $U = \cup_{j=1}^p U_j$. Now we choose $r > 0$ so small that for every $1 \le j \le p$ the open set

$$\widetilde{U}_j = \{(\zeta, z) : \zeta \in U_j, \rho(\widetilde{f}(\zeta), z) < r\} \subset\subset U_j \times B_j.$$

Let $X = \{(\zeta, z) : \zeta \in U, \rho(\widetilde{f}(\zeta), z) < r\}$ and let $P(\zeta, z) = \zeta$ be a projection of X onto U. Clearly, $\Gamma^f_K \subset X \subset X'$.

Our next step is to choose appropriate plurisubharmonic functions on $\widetilde{U}_j$. For this we fix $\varepsilon > 0$ whose precise value will be determined later. For each $1 \le j \le p$ we take a holomorphic mapping g_j of a neighborhood of $\overline{U}_j \cap K$ into M such that $\rho(g_j(\zeta), f(\zeta)) < \varepsilon$ on $\overline{U}_j \cap K$. There is a neighborhood $W_j \subset Y$ of $\overline{U}_j \cap K$ such that g_j is defined on W_j and $\rho(g_j(\zeta), \widetilde{f}(\zeta)) < \varepsilon$ on W_j and g_j maps W_j into B_j. Let $u_j(\zeta, z) = \max\{\log \|\phi_j(z) - \phi_j(g_j(\zeta))\|, \log \varepsilon\}$ be the functions on $\widetilde{W}_j = P^{-1}(W_j)$.

The functions u_j have the following properties. If $(\zeta, z) \in \partial X$ and $\rho(z, \widetilde{f}(\zeta)) = r$, then

$$u_j(\zeta, z) \ge \log(a^{-1}\rho(z, g_j(\zeta))) \ge \log(a^{-1}(r - \varepsilon)).$$

Suppose that $(\zeta, z) \in \widetilde{U}_j \cap \widetilde{U}_l$. Then

$$\begin{aligned} \|\phi_l(z) - \phi_l(g_l(\zeta))\| &\le \|\phi_l(z) - \phi_l(g_j(\zeta))\| + \|\phi_l(g_j(z)) - \phi_l(g_l(\zeta))\| \\ &\le \|\phi_l(z) - \phi_l(g_j(\zeta))\| + 2a\varepsilon \le a^2\|\phi_j(z) - \phi_j(g_j(\zeta))\| + 2a\varepsilon. \end{aligned}$$

Thus if $\|\phi_j(z) - \phi_j(g_j(\zeta))\| < \varepsilon$, then

$$u_l(\zeta, z) - u_j(\zeta, z) \le \log \varepsilon + \log(a^2 + 2a) - \log \varepsilon = \log(a^2 + 2a) = b.$$

If $\|\phi_l(z) - \phi_l(g_l(\zeta))\| < \varepsilon$, then

$$u_l(\zeta, z) - u_j(\zeta, z) \le \log \varepsilon - \log \varepsilon = 0.$$

If $\|\phi_j(z) - \phi_j(g_j(\zeta))\| \ge \varepsilon$ and $\|\phi_l(z) - \phi_l(g_l(\zeta))\| \ge \varepsilon$, then

$$u_l(\zeta, z) - u_j(\zeta, z) \le \log(a^2 + 2a) = b.$$

Thus

$$\|u_j - u_l\|_{\widetilde{U}_j \cap \widetilde{U}_l} \le b.$$

Finally, if $\zeta \in W_j$ then $u_j(\zeta, \widetilde{f}(\zeta)) \le \log(a\varepsilon)$.

By Theorem 2.1 there is a constant $c > 0$ which does not depend on ε, a neighborhood Z, which we may assume to be Stein, of K and a plurisubharmonic function u on $\widetilde{Z} = P^{-1}(Z)$ such that

$$\|u - u_j\|_{\widetilde{W}_j \cap \widetilde{Z}} \le cb.$$

Now if $\zeta \in Z$ and $\rho(z, f(\zeta)) = r$, then
$$u(\zeta, z) \geq \log(a^{-1}(r-\varepsilon)) - cb,$$
while if $z = \widetilde{f}(\zeta)$ then
$$u(\zeta, z) \leq cb + \log(a\varepsilon).$$
So if we take $\varepsilon > 0$ so small that $\log(a^{-1}(r-\varepsilon)) > \log(a\varepsilon) + 2cb$, then the open set
$$D = \{(\zeta, z) \in \widetilde{Z} : u(\zeta, z) < \log(a^{-1}(r-\varepsilon))\}$$
contains $\Gamma_Z^{\widetilde{f}}$ and $u(\zeta, z) = \log(a^{-1}(r-\varepsilon))$ when $(\zeta, z) \in \partial D$ and $\zeta \in Z$. Hence there is a plurisubharmonic function v on D such that $v(\xi, z) \to \infty$ as (ξ, z) approaches a point (ζ, z), $(\zeta, z) \in \partial D$ and $\zeta \in Z$.

Finally, we take a plurisubharmonic exhaustion function ψ on Z and let $v(\zeta, z) = u(\zeta, z) + \psi(\zeta)$. This is a plurisubharmonic exhaustion function on D.

A theorem of R. Narasimhan (see [Nar] and [AnNar]) claims that a manifold is Stein if it has a plurisubharmonic exhaustion function and a strictly plurisubharmonic function. Let us construct the latter function on X. For this on each $\widetilde{W}_j$ we take the functions $u_j(\zeta, z) = \|\phi_j(z)\|^2$ and apply Corollary 2.2. This implies that D is a Stein manifold.

A Stein neighborhood of K and the parameter r can be chosen as small as we want and this shows that K has a basis of Stein neighborhoods in $N \times M$. $\square$

This theorem has an immediate corollary.

Corollary 3.2. *If in assumptions of Theorem 3.1 $f \in A(K, M)$ and, additionally, $A(K, M)$ has the local Mergelyan property, then the graph of f on K has a basis of Stein neighborhoods in $N \times M$.*

4. Applications

Throughout this sections we assume that M is a complex manifold with a Riemann metric ρ_M and K is a compact set with a basis of Stein neighborhoods in a complex manifold N.

The first application allows us to construct Stein neighborhoods for special products.

Theorem 4.1. *If $K' = K \times [0,1] \subset N' = N \times \mathbb{C}$ and $f(z,t), z \in K, t \in [0,1]$, is a continuous mapping of K' into M such that $f(\cdot, t) \in H(K, M)$ for each $t \in [0,1]$, then $f \in H_{loc}(K', M)$ and $\Gamma_{K'}^f$ has a basis of Stein neighborhoods in $N' = N \times \mathbb{C} \times M$.*

Proof. Clearly, K' has a basis of Stein neighborhoods in $N \times \mathbb{C}$. By Theorem 3.1 all we need to prove is that $f \in H_{loc}(K', M)$. For this let us take any point $z_0 \in K$ and consider the compact set $K'_0 = \{z_0\} \times [0,1] \subset N'$. Clearly, $H_{loc}(K'_0, M) = C(K'_0, M)$ and, therefore, by Theorem 3.1 the graph $\Gamma_{K'_0}^f$ has a Stein neighborhood Y in $N' \times M$. Let us denote by Y_1 a neighborhood of $\Gamma_{K'_0}^f$ which compactly belongs to Y. There is a neighborhood V of z_0 in N such that $\Gamma_{V'}^f \subset\subset Y_1$, where $V' = V \times [0,1]$.

Let F be an imbedding of Y into $\mathbb{C}^p$ as a complex submanifold. By [GR, Theorem 8.C.8] there are an open neighborhood U of $F(\overline{Y}_1)$ in $\mathbb{C}^p$ and a holomorphic retraction P of U onto $F(Y)$. We set $\hat{f}(z,t) = F(z,t,f(z,t))$ for $(z,t) \in \overline{V}'$.

Let us show that the function $\hat{f}$ can be uniformly approximated with any precision on $\overline{V}' \cap K'$ by holomorphic mappings defined on the neighborhoods of $\overline{V}' \cap K'$. For this for $\varepsilon > 0$ and for each $t \in [0,1]$ we can find ε_t between 0 and ε and a neighborhood Z_t of K and a mapping $f_t \in A(Z_t, M)$ such that $\rho_M(f_t(z), f(z,t)) < \varepsilon_t$ when $z \in K$ and $(z, t, f_t(z)) \in Y$ when z is in a neighborhood W_t of $\overline{V} \cap K$. For $z \in W_t$ we set $\hat{f}_t(z) = F(z, t, f_t(z))$. There is a constant C depending only on Y_1 and F such that $\|\hat{f}_t(z,t) - \hat{f}(z,t)\| < C\varepsilon$ when $z \in W_t \cap K$.

Thus we can select points $0 = s_0 < s_1 < \cdots < s_q = 1$ and $t_j \in [s_{j-1}, s_j]$, $1 \le j \le q$ such that $\|\hat{f}_{t_j}(z) - \hat{f}(z,t)\| < 2C\varepsilon$ when $t \in [s_{j-1}, s_j]$ and $z \in W \cap K$. Set $W = \cap_{j=1}^{q} W_{t_j}$. We define a mapping $\hat{g}$ of $W' = W \times [0,1]$ into $\mathbb{C}^p$ as follows: in the first step we set $\hat{g}(z,t) = \hat{f}_{t_j}(z)$ when $t \in [s_{j-1}, s_j)$ and in the second step we choose intervals $[x_j, y_j]$ around point s_j, $j = 1, \dots, q-1$, and set

$$\hat{g}(z,t) = \frac{y_j - t}{y_j - x_j} \hat{f}_{t_{j-1}}(z) + \frac{t - x_j}{y_j - x_j} \hat{f}_{t_j}(z)$$

on $[x_j, y_j]$. Choosing these intervals small we can require that $\|\hat{g}(z,t) - \hat{f}(z,t)\| < 3C\varepsilon$.

Since $\hat{g}$ is a continuous mapping of W' we can use the Stone–Weierstrass theorem to approximate it uniformly on $\overline{V}' \cap K'$ by holomorphic mappings g_l which are polynomials of degree at most $2l$ in t. For example we can use the convolution formula (51) from the proof of Theorem 7.26 in [Ru]. The obtained functions $\hat{g}_l$ will be defined on $W \times \mathbb{C}$ and the direct verification of the proof in [Ru] shows that they holomorphic in z and converge uniformly to $\hat{g}$ as $l \to \infty$.

Picking up $\varepsilon > 0$ sufficiently small and the approximation above sufficiently precise we can be assured that $\hat{g}_l$ maps some neighborhood of $\overline{V}' \cap K'$ into Y and $\|\hat{f}(z,t) - P(\hat{g}_l(z,t))\| < 4C\varepsilon$ on $\overline{V}' \cap K'$.

Let $g_l = \Pi \circ F^{-1} \circ P \circ \hat{g}_l$, where Π is the projection $N \times M$ onto M. Due to the continuity of $\Pi \circ F^{-1}$ choosing ε small we guarantees that g_l uniformly approximates f with any prescribed precision. □

In the results below we will use polynomials taking prescribed values at given points. While on $\mathbb{C}$ this job is done by Lagrange interpolating polynomials, there is no a canonical procedure in several variables. To avoid ambiguity we will use the following simple result.

Lemma 4.2. *Suppose that a finite set $\sigma \subset \mathbb{C}^q$ of distinct points $\{\sigma_1, \dots, \sigma_q\}$ and a set $h = \{h_1, \dots, h_q\} \subset \mathbb{C}$. Let $\|h\| = \max_{1 \le j \le q} |h_j|$. Then for every $R > 0$ there is a constant C depending only on σ such that there is a polynomial $L_{\sigma,h}$ of degree at most q on $\mathbb{C}^q$ taking values h_j at all points σ_j and whose uniform norm does nor exceed $C\|h\|$ when $\|z\| \le R$.*

Proof. Let us fix q hyperplanes defined as solutions of the equations $L_j(z) = \langle a_j, z \rangle + b_j = 0$ with $\|a_j\| = 1$ such that $L_j(\sigma_i) = 0$ if and only if $i = j$. Set

$$L_{\sigma,h}(z) = \sum_{k=1}^{q} h_k \frac{\prod_{j \ne k} L_j(z)}{\prod_{j \ne k} L_j(\sigma_k)}.$$

It is easy to see that $L_{\sigma,h}$ has all required properties. □

The following application is the Mergelyan-type approximation of mappings in $A(K, M)$. When $K \subset \mathbb{C}$ this problem was studied in [Cha].

Theorem 4.3. *Suppose that $\sigma = (\sigma_1, \dots, \sigma_q\}$ is a finite set in K, $h = \{h_1, \dots, h_q\}$ is a set in M and $A(K, \mathbb{C})$ has the Mergelyan property. For every $f \in H_{loc}(K, M)$ and for every $\varepsilon > 0$ there is $\delta > 0$ such that if $\rho_M(f(\sigma_j), h_j) < \delta$, $1 \le j \le q$, then one can find a neighborhood U of K and $g \in A(U, M)$ such that $\rho_M(f(\zeta), g(\zeta)) < \varepsilon$ for all $\zeta \in K$ and $g(\sigma_j) = h_j$, $1 \le j \le q$.*

Proof. Since $f \in H_{loc}(K, M)$, by Theorem 3.1 the graph Γ^f_K has a Stein neighborhood Y in $N \times M$. Let F be an imbedding of Y into $\mathbb{C}^p$ as a complex submanifold. By [GR, Theorem 8.C.8] there are an open neighborhood $U \subset\subset \mathbb{C}^p$ of $F(\Gamma^f_K)$ in $\mathbb{C}^p$ and a holomorphic retraction P of U onto $F(Y)$.

Let $\hat{f}(\zeta) = F(\zeta, f(\zeta))$. Since $A(K, \mathbb{C})$ has the Mergelyan property for any $\eta > 0$ there is a neighborhood V of K and $\hat{g}_1 \in A(V, \mathbb{C}^p)$ such that $\|\hat{f}(\zeta) - \hat{g}_1(\zeta)\| < \eta$ on K and $\hat{g}_1(V) \subset\subset U$. Choose $\delta > 0$ such that $\|F(\sigma_j, f(\sigma_j)) - F(\sigma_j, h_j)\| < \eta$, $1 \le j \le q$. Then $\|\hat{g}_1(\sigma_j) - F(\sigma_j, h_j)\| < 2\eta$. By Lemma 4.2 the uniform norm on U of an interpolating polynomial mapping $L : \mathbb{C}^p \to \mathbb{C}^p$ such that $L(\hat{g}_1(\sigma_j)) = F(\sigma_j, h_j) - \hat{g}_1(\sigma_j)$ does not exceed $c\eta$ on U, where c depend only on U and σ. If $\hat{g}(\zeta) = \hat{g}_1(\zeta) + L(\hat{g}_1(\zeta))$ then $\hat{g}(\sigma_j) = F(\sigma_j, h_j)$ and $\|\hat{f}(\zeta) - \hat{g}(\zeta)\| < (1 + c)\eta$ on K.

Define $g = \Pi \circ F^{-1} \circ P \circ \hat{g}$. Due to the continuity of $\Pi \circ F^{-1} \circ P$ choosing η sufficiently small guarantees that g has all needed properties. □

This theorem has an interesting corollary which was proved by E. L. Stout in [St] when $A(K, \mathbb{C}) = C(K, \mathbb{C})$.

Corollary 4.4. *If $A(K, \mathbb{C})$ has the Mergelyan property then $A(K, M)$ has the Mergelyan property.*

Proof. If $A(K, \mathbb{C})$ has the Mergelyan property then $A(K, M)$ has the local Mergelyan property and by the previous result $H_{loc}(K, M) = H(K, M)$. Hence $A(K, M) = H(K, M)$. □

To give another application we choose a Riemann metrics ρ_N on N and introduce a topology on the set $\mathcal{S}(N, M) = \mathcal{S}$ of all pairs (K, f), where K is a compact set in N and $f \in A(K, M)$. Given any continuous extension Φ of f to N and $\varepsilon > 0$ we define a Φ, ε-neighborhood of (K, f) as a set of all pairs $(L, g) \in \mathcal{S}$ such that the Hausdorff distance $\rho(K, L)$ between K and L is less than ε and $\rho_M(\Phi(\zeta), g(\zeta)) < \varepsilon$ for all $\zeta \in L$.

It is easy to verify that if U is a (Φ, ε)-neighborhood of (K, f), V is a (Ψ, δ)-neighborhood of (L, g) and $(N, h) \in U \cap V$, then there is a neighborhood of (N, h) lying in $U \cap V$. Hence our choice of neighborhoods defines a topology on $\mathcal{S}$.

Suppose that we have a family of compact sets $L_t \subset N$, $0 \le t \le 1$, and a finite set $\sigma \subset L_t$ for all t. Such a family is *Radó continuous with respect to σ* if there are homeomorphisms $\phi_t \in A(L_0, N)$ mapping L_0 onto L_t such that for every $t \in [0, 1]$ the set σ belongs to the set of fixed points of ϕ_t and the mappings $\phi_s \circ \phi_t^{-1}$ converge uniformly to identity as $s \to t$.

In [Ra] T. Radó gave necessary and sufficient conditions for a family L_t of closures of connected and simply connected domains in $\mathbb{C}$ to be Radó continuous. For us this notion is useful because, evidently, if a family L_t is Radó continuous and $g_0 \in A(L_0, M)$, then the path $(L_t, g_0 \circ \phi_t^{-1})$ is continuous in the topology introduced above.

The following theorem shows that under reasonable assumptions Φ, ε-neighborhoods are "convex".

Theorem 4.5. *Suppose that $f \in H(K, M)$. For every $\varepsilon > 0$ there is $\delta > 0$ with the following properties: if:*

(1) *$\sigma = \{\sigma_1, \dots, \sigma_q\}$ is a set of distinct points in N;*
(2) *L_t, $0 \le t \le 1$, is a family of compact sets $L_t \subset N$ lying in the ρ_N, δ-neighborhood of K such that $\sigma \subset L_t \cap K$ when $0 \le t \le 1$;*
(3) *the family L_t is Radó continuous with respect to σ;*
(4) *mappings $g_0 \in A(L_0, M)$ and $g_1 \in A(L_1, M)$ and there is a continuous extension Φ of f to N such that $\rho_M(g_0(\zeta), \Phi(\zeta)) < \delta$ and $\rho_M(g_0(\zeta), \Phi(\zeta)) < \delta$ when $\zeta \in L_0$ or $\zeta \in L_1$ respectively;*
(5) *$g_0(s_j) = g_1(s_j)$, $1 \le j \le q$,*

then there is a continuous path (L_t, g_t), $0 \le t \le 1$, such that $\rho_M(g_t(\zeta), \Phi(\zeta)) < \varepsilon$ and $g_t(\sigma_j) = g_0(\sigma_j)$, $1 \le j \le q$ and $0 \le t \le 1$.

Proof. By Theorem 3.1 the graph Γ_K^f has a Stein neighborhood Y in $N \times M$. Let F be an imbedding of Y into $\mathbb{C}^p$ as a complex submanifold. By [GR, Theorem 8.C.8] there are an open neighborhood $U \subset\subset \mathbb{C}^p$ of $F(\Gamma_K^f)$ in $\mathbb{C}^p$ and a holomorphic retraction P of U onto $F(Y)$. If Ψ is a mapping of $A \subset N$ into M then we set $\hat{\Psi}(\zeta) = F(\zeta, \Psi(\zeta))$, $\zeta \in A$.

By Theorem 4.3 for any $\eta > 0$ there is a neighborhood D of K and $h \in A(D, M)$ such that $\|\hat{h}(\zeta) - \hat{\Phi}(\zeta)\| < \eta$ when $\zeta \in D$, $\Gamma_D^h \subset Y$ and $F(\Gamma_D^{\hat{h}}) \subset U$. Moreover, we can find h such that $h(\sigma_j) = g_0(\sigma_j)$ provided $\rho_M(f(\sigma_j), g_0(\sigma_j))$ doess not exceed some $\delta > 0$ for all j.

We may assume that δ is so small that all $L_t \subset D$. Let $\hat{h}_0$ and $\hat{h}_1$ be the restrictions of $\hat{h}$ to L_0 and L_1 respectively. Let $\hat{d}_0(\zeta) = \hat{h}_0(\zeta) - \hat{g}_0(\zeta)$ and $\hat{d}_1(\zeta) = \hat{h}_1(\zeta) - \hat{g}_1(\zeta)$. Then

$$\|\hat{d}_0(\zeta)\| \le \|\hat{h}(\zeta) - \hat{f}(\zeta)\| + \|\hat{f}(\zeta) - \hat{g}_0(\zeta)\| \le \eta + \|\hat{f}(\zeta) - \hat{g}_0(\zeta)\|$$

for $\zeta \in L_0$ and a similar inequality holds for $\hat{d}_1$. Reducing if needed the size of δ we may assume that $\|\hat{d}_0\|$ on L_0 and $\|\hat{d}_1\|$ on L_1 do not exceed 2η.

On L_t we define

$$\hat{b}_t(\zeta) = (1-t)\hat{d}_0(\phi_t^{-1}(\zeta)) + t\hat{d}_1(\phi_1(\phi_t^{-1}(\zeta))),$$

where $\phi_t \in A(L_0, N)$ are homeomorphisms of L_0 onto L_t keeping σ fixed. Clearly, $\|b_t(\zeta\| < 2\eta$ for all t and $\zeta \in L_t$, $\hat{b}_0 = \hat{d}_0$, $\hat{b}_1 = \hat{d}_1$ and $\hat{b}_t(\sigma_j) = 0$. By an observation preceding the statement of the theorem the path $(L_t, \hat{b}_t)$ is continuous in $\mathcal{S}(N, \mathbb{C}^p)$.

Let $\hat{g}_t(\zeta) = \hat{h}(\zeta) - \hat{b}_t(\zeta)$. Then $\hat{g}_t(\sigma_j) = \hat{g}_0(\sigma_j)$ and

$$\|\hat{g}_t(\zeta) - \hat{\Phi}(\zeta)\| \le \|\hat{h}(\zeta) - \hat{\Phi}(\zeta)\| + 2\eta \le 3\eta.$$

Define $g_t = \Pi \circ F^{-1} \circ P \circ \hat{g}_t$. Due to the continuity of $\Pi \circ F^{-1} \circ P$ choosing η sufficiently small guarantees that g_t has all needed properties. $\square$

One of the essential problem in complex geometry is moving objects preserving their complex structures. The problem was investigated in [Gr], where the notion of sprays was introduced. Later this notion obtained a significant development in papers of Forstnerič and his coauthors. We will use the definition of sprays close to one given in [DrnFor].

Let $f \in A(K, M)$. A spray with an exceptional set $\sigma = \sigma(f) \subset K$ is a map $F : K \times V \to M$, where V (the parameter set of F) is an open subset of an Euclidean space $\mathbb{C}^N$ containing the origin, such that the following holds:

(1) F is in $A(K \times V, M)$;
(2) $F(z,0) = f(z)$ and $F(z,t) = f(z)$ for $z \in \sigma$ and $t \in V$;
(3) for every $z \in K \setminus \sigma$ the map $\partial_t F(z,t) : \mathbb{C}^N \to T_{F(z,t)}M$ is surjective (the domination condition).

The following theorem establishing an existence of sprays can be proved exactly as Lemma 4.2 in [DrnFor] where it was established for compact sets with smooth boundary. For the sake of completeness we give another proof here.

Theorem 4.6. *Let $f \in H_{loc}(K, M)$ and $\sigma = \{\sigma_1, \dots, \sigma_k\}$ be a finite set in K. Then there is a spray G with the exceptional set σ and $G(z,0) = f(z)$ on K.*

Proof. We take a Stein neighborhood Y of Γ_K^f in $N \times M$ and imbed it into $\mathbb{C}^p$ as a complex submanifold by a holomorphic mapping G. Let $\hat{\sigma}_j = F(\sigma, f(\sigma_j))$ and $\hat{\sigma} = \{\hat{\sigma}_1, \dots, \hat{\sigma}_k\}$. There are exactly p polynomials $P_1, \dots, P_N$ on $\mathbb{C}^p$ such that $\hat{\sigma} = \{P_1 = \cdots = P_p = 0\}$. This can be proved by induction in p. If $p = 1$ then P_1 is a polynomial of degree k with simple roots in $\hat{\sigma}$. If it is proved for $p-1$ then we represent $\mathbb{C}^p$ as $\mathbb{C}^{p-1} \times \mathbb{C} = \{(z', \zeta) : z' \in \mathbb{C}^{p-1}, \zeta \in \mathbb{C}\}$. We may assume that no two points in $\hat{\sigma}$ have the same coordinates in $\mathbb{C}^{p-1}$. Let $\hat{\sigma}' = \{\hat{\sigma}'_1, \dots, \hat{\sigma}'_k\}$ be the projection of $\hat{\sigma}$ onto $\mathbb{C}^{p-1}$. By the induction of assumptions there are polynomials $P_1, \dots, P_{p-1}$ on $\mathbb{C}^{p-1}$ such that $\hat{\sigma}' = \{P_1 = \cdots = P_{p-1} = 0\}$. Denote by $\hat{\sigma}''_j$ the projection of $\hat{\sigma}_j$ on $\mathbb{C}$ and take an interpolating polynomial Q_p on $\mathbb{C}^{p-1}$ such that $Q_p(\hat{\sigma}'_j) = \hat{\sigma}''_j$. Let $P_p(z', \zeta) = \zeta - Q_p(z')$. Clearly, $\hat{\sigma} = \{P_1 = \cdots = P_p = 0\}$.

There are an open neighborhood U of $F(\Gamma_K^f)$ in $\mathbb{C}^p$ and a holomorphic retraction P of U on $G(Y)$. We may assume that U is so small that dP has the maximal rank on U. There is $\varepsilon > 0$ so small that if all $v_j \in \mathbb{C}^N$, $1 \le j \le N$, have the norm less than ε and $v = (v_1, \dots, v_N)$, then

$$H(\zeta, v) = F(\zeta, f(\zeta)) + P_1(\zeta, f(\zeta))v_1 + \cdots + P_N(\zeta, f(\zeta))v_N \in U$$

for all $\zeta \in K$. Let Π be the natural projection of $M \times N$ onto M. Hence the mapping $G(\zeta, v) = \Pi \circ F^{-1} \circ P \circ H(\zeta, v)$ is well defined when v is in the ball of radius ε in $\mathbb{C}^{p^2}$, holomorphic, $G(\zeta, 0) = f(\zeta)$ if $\zeta \in \sigma$ and for every $\zeta \in K \setminus \sigma$ the map $\partial_v G(\zeta, v) : \mathbb{C}^{p^2} \to T_{G(\zeta,v)}M$ is surjective. To show the last statement we assume that $P_1(\zeta, f(\zeta)) \neq 0$ and let $v_2 = \cdots = v_N = 0$. Then it is easy to see that $\partial_{v_1} G(\zeta, v_1) : \mathbb{C}^{p^2} \to T_{G(\zeta,v_1)}M$ is surjective because $\partial_{v_1} H$, dP and $d(\Pi \circ (F^{-1})$ are surjective. $\square$

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